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1 Introduction

I am an applied mathematician and computer scientist who is currently transitioning into data science. I completed my Ph.D. in Mathematics, with a minor in computational science, at Penn State University in summer 2019 under the direction of Dr. Diane Henderson. In this research statement, I will first describe the research I participated in during my time as an undergraduate. Next, I will briefly introduce my thesis research, which focused on the mathematical modeling of complex fluid interactions involving fluid dynamics, numerical simulations, ordinary and partial differential equations, and parametric studies. Finally, I will conclude with an outline of my current and future directions of study.

2 Undergraduate Research

I began my research career as a research assistant in the Mathematical Physics Research Group at Lebanon Valley College under the direction of Dr. David Lyons and Dr. Scott Walck. This work was done as part of an RUI (Research in Undergraduate Institutions) in Quantum Information. Quantum entanglement is an important component in the implementation of quantum information processing tasks, such as quantum computing algorithms and quantum communications protocols. Thus, understanding entanglement is an important problem in quantum information science. My work focused on the quantum entanglement of 4-qubit states. I wrote and used GAP code to find examples of qubit states with the desired entanglement and examined 4-qubit states that share a particular type of entanglement. I proved that only those states with a particular structure had the desired type of entanglement. In addition, I proved the following:

(2.1) Theorem. *An n-qubit state vector ψ has the maximum stabilizer dimension minus one if and only if it is a product of $\frac{n-2}{2}$ singlets together with two unentangled qubits (for n even) or a product of $\frac{n-3}{2}$ singlets together with three unentangled qubits (for n odd).*

For more information about my quantum entanglement research, please see [6].

3 Thesis Research

The generation of ocean waves by wind has long been a topic of interest. However, it has only been in the past 100 years or so, starting with work by Jeffreys [3, 4], that significant progress has been made on this topic. Despite this progress, there is still much that is not understood. For example, the predictions of the growth rate are typically underestimates by an order of magnitude [2]; the role of the interfacial current at the air-water interface that is set up by the wind is neglected in all studies of which we are aware; and the wind is considered to be steady in every study of which we are aware. Thus, in my dissertation, the wind (i) is considered to be time-dependent; (ii) sets up an interfacial current; and (iii) is an exact solution to the Navier-Stokes equations [1]. Our goal is to predict the growth rates of waves with a given wavenumber that arise as perturbations to this base flow.

The problem formulation of partial differential equations for the perturbations (the waves) is simplified by setting time to be a parameter, allowing for a Fourier decomposition in time as well as in the propagation distance. The resulting ordinary differential equations are with respect to the vertical spatial variable and are of Airy-type. The differential equations are solved exactly to reduce the problem to a linear, algebraic system of 19 equations that has a vanishing determinant for nonzero waves. The result is a relationship between the complex frequency and real wavenumber. This relationship is analyzed numerically to determine the oscillation frequency corresponding to the largest growth rate for a given wavenumber. Parametric studies, in which the interfacial current, the time, and the wavenumber are varied, show that for very small times the interfacial current plays an important role in this maximum growth rate frequency. Further, a bound on time is found for which the base flow model considered becomes inadequate.

The emphasis in my dissertation was the growth rate of an instability of the interface between air flowing over water. However, the theory developed is generic to any two immiscible, hydrostatically stable fluids in a gravitational field and may thus be used to obtain the growth rates of interfacial waves in any system comprising fluids of arbitrary densities and viscosities.

3.1 Problem Set-Up

We consider the horizontal flow of two immiscible, incompressible fluids with kinematic viscosities, ν_j, and densities, ρ_j, where $j = 1, 2$ correspond to the upper and lower fluids, respectively. Here, we are interested in air over water, but the theory is generic to any two immiscible, hydrostatically stable fluids in a gravitational field. A schematic of the domain is shown in Figure 1. It is two-dimensional with position vector, $\mathbf{x} = (x, 0, z)$, $x, z \in (-\infty, \infty)$ and velocity vector $\mathbf{q}_j(x, z, t) = \{u_j(x, z, t), 0, w_j(x, z, t)\}$. The location of $x = 0$ is arbitrary; $z = 0$ is at the quiescent interface.

3.2 Base Flow for Two Fluids

We first find the base flow for the two fluid problem, where there is no perturbation due to waves. By solving a heat equation in both fluids (subject to boundary conditions that account for the continuity of tangential and normal stress across the interface and that the tangential component of velocity must be continuous across the interface to avoid infinite shear stresses there) we find that the velocity U_i $(i = 1, 2)$ is given by

$$U_1(\zeta_1) = U^\infty \left(\frac{2}{\sqrt{\pi}\left(1 + R\sqrt{V}\right)} \int_0^{\zeta_1} e^{-\xi^2} d\xi + \frac{V_0}{U^\infty} \right), \tag{3.1a}$$

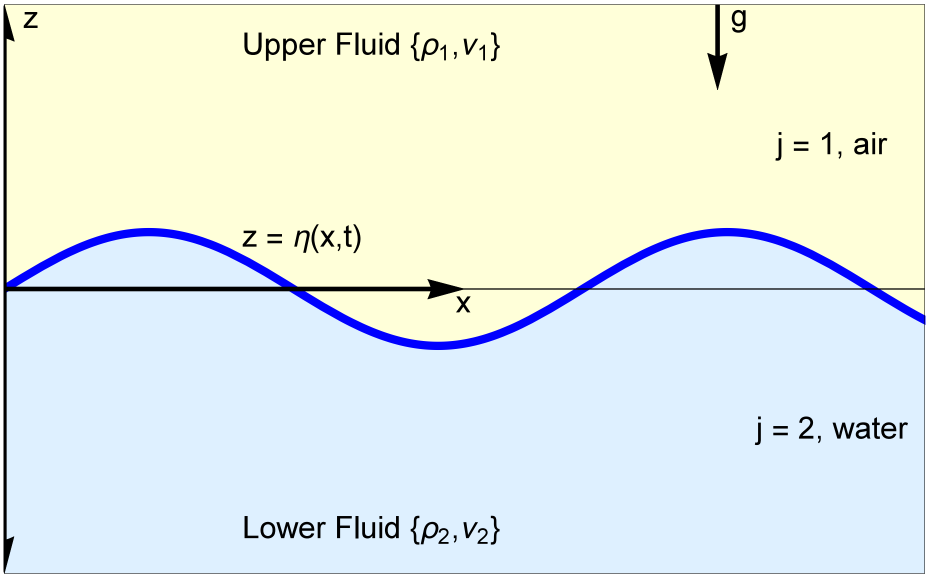


Figure 1: Schematic of the fluid domain

$$U_2(\zeta_2) = U^\infty \left(\frac{2R\sqrt{V}}{\sqrt{\pi}\left(1+R\sqrt{V}\right)} \int_0^{\zeta_2} e^{-\xi^2} d\xi + \frac{V_0}{U^\infty} \right), \tag{3.1b}$$

where

$$V_0 := U^\infty \left(\frac{R\sqrt{V}}{1+R\sqrt{V}} \right) \tag{3.2}$$

is the interfacial current, which is independent of time, and $R := \dfrac{\rho_1}{\rho_2}, V := \dfrac{\nu_1}{\nu_2}$ are non-dimensional quantities.

Interfacial waves enter this problem as a perturbation to the base flow. However, we have been unable to obtain stability results for the full solution (3.1) due to its unsteady nature and z-dependence. Instead, we will consider the stability of an approximate base flow in z, motivated by the observation of the two-region nature of the solution. Here, we will consider t to be a parameter, and will approximate the z-dependence of the base flow with the first two terms in a Taylor series expansion, so that

$$U_1(z,t) \approx \begin{cases} U^\infty, & z_1 \le z \\ s_1 z + V_0, & 0 \le z \le z_1 \end{cases}, \tag{3.3a}$$

$$U_2(z,t) \approx \begin{cases} s_2 z + V_0, & z_2 \le z \le 0 \\ 0, & z \le z_2 \end{cases}. \tag{3.3b}$$

Here, z_1 and z_2 are reference distances.

3.3 Perturbed Flow for Two Fluids

To study the stability of the base flow, we perturb it using asymptotic series and then take the curl of the linearized Navier-Stokes equations to obtain a diffusion equation with non-constant coefficients for the vorticity, $\Omega = \nabla \times \mathbf{q}$, which corresponds to

$$\frac{\partial \Omega_{jl}}{\partial t} + U_j \frac{\partial \Omega_{jl}}{\partial x} - \left(\frac{\partial \phi_{jl}}{\partial z} + \frac{\partial A_{jl}}{\partial x} \right) \frac{\partial^2 U_j}{\partial z^2} = \nu_j \nabla^2 \Omega_{jl}, \tag{3.4}$$

where

$$\Omega_{jl} := \nabla^2 A_{jl}. \tag{3.5}$$

To obtain explicit solutions, we now approximate the base flow using (3.3) and consider t to be a parameter. A consequence of using the approximate base flow is that $(\partial^2 U_j) / (\partial z^2) = 0$. As a result, the influence of the irrotational part of the flow in (3.4) disappears.

3.4 Numeric Results

Once we had computed the base flow solution, we stated and solved the linearized boundary-value problem for the general solution of the perturbed system under the onset of waves. Specifically, we found representations for the vorticity, the potential and stream functions, the velocity components, and the pressure. This resulted in 19 unknowns, which required 19 equations to determine the entire system. We substituted the solutions for velocities and pressure in the boundary conditions and other constraints to obtain a 19×19 system of linear, algebraic equations. Ideally, we wanted to determine the growth rate explicitly as a function of the physical parameters of our system. However, the set of 19 equations, while linear in the unknowns, were sufficiently complicated with respect to the complex frequency that we were forced to analyze the system numerically.

To gain insight into the behavior of the coupled fluid system, we began our analysis by nondimensionalizing the linear equations. Before analyzing the linear system numerically, we had to fix the reference distance where the transition between linear and constant wind profiles in the upper fluid (air) occurred. However, this could not be measured exactly in the laboratory. In our analysis, we considered two choices for the reference distance - one based on the laboratory measurements and one based on Miles' [7] log-law for the wind and the friction velocity near the interface (which depends on the interfacial current, in our problem formulation). Our last step before continuing with the numerical analysis was to choose a value for the interfacial current. We saw that this was not necessarily straight-forward because the measured interfacial current did not match the predicted value. We considered both cases.

Our main goal was to determine which wavenumber produced the largest growth rate for the waves. To this end, we computed the rank of the 19×19 coefficient matrix, recognizing that the system has rank 18 in the physically relevant case (see Figure 2 for an example of our output). To compute the matrix rank, we used Mathematica and formed the matrix using the non-dimensional equations. The following is a listing of our main conclusions:

- There are no rank 18 solutions (with positive growth rates) when the oscillation frequency is close to the free-wave value.

- The oscillation frequency for maximum growth rate tends to increase with increasing wavenumber (decreasing wavelength) for the smaller time case while increasing and then decreasing for the longer time case, when the measured interfacial current value is used.

- The oscillation frequency for maximum growth rate is proportional to kV_0 at the same time when the measured interfacial current is used. The proportionality constant is 1/2. The oscillation frequency for maximum growth rate is (close to) kV_0 at the small time when the predicted interfacial current is used. The predicted and measured interfacial currents differ by an order of magnitude with the lower value corresponding to a smaller time after the wind begins to blow. These results provide evidence that the interfacial current plays a significant role in the growth of wind waves at very short times.

For the results presented in my dissertation, we concluded that the interfacial current plays an important role in the development of wind-generated wavefields, for small time.

4 Current and Future Research

What follows are problems that offer potential avenues for future research, based off my thesis research.

- **Sea Breezes** A sea breeze (or onshore breeze) is a gentle wind that develops over the ocean because the morning sun heats the land more quickly than the water. Understanding the development and strength of sea breezes is important in meteorology. I have found the analytic solution for the base flow in this problem and will use similar analytic and numeric techniques to model the perturbed flow.

- **Non-Clean Interface** In my thesis research, I assumed that the interface between the two fluids was clean. However, this is not the case in the ocean - it is likely that there are bubbles or some debris on the surface of the ocean. This impacts the density of the fluids because instead of having two distinct densities, the density is going to depend on the height z (and possibly on the location x). During the 2014 - 2015 school year at Penn State, I mentored an undergraduate student who began work on this problem.

Though I enjoyed my thesis research, I have begun to turn to research in the area of data science. At my current institution, Lebanon Valley College, we have strong majors in both Computer & Data Science and Actuarial Science. I have been tasked, along with my colleague Dr. Wanying Fu, with growing our undergraduate research in these areas. To this end, we are exploring the application of both PDEs and machine learning models to topics in finance. Some potential avenues for future research follow.

- **Nonlinear Wave Interactions** As we saw from my thesis research, there is still much that is not understood when considering the generation of ocean waves by wind. Even in the linearized case that I considered, we were unable to determine the growth rate explicitly as a function of the physical parameters of our system. Increasing the complexity of the problem by considering nonlinear wave interactions, it is unlikely that an analytic solution can be found. However, it is possible that neural networks and deep learning techniques can be used to provide accurate estimates of nonlinear interactions for wind wave spectra.

- **Finance** With Dr. Fu, I am beginning to explore the use of deep learning techniques for problems in financial prediction (for example, in designing and pricing securities, constructing portfolios, and risk management) and classification.

Both of these topics are accessible to undergraduates who desire to do research.

References

[1] Csanady, G. *Air-Sea Interaction, Laws and Mechanisms.* No. pg. 5. Cambridge University Press, 2001.

[2] Janssen, P. *The Interaction of Ocean Waves and Wind.* Cambridge University Press, 2004.

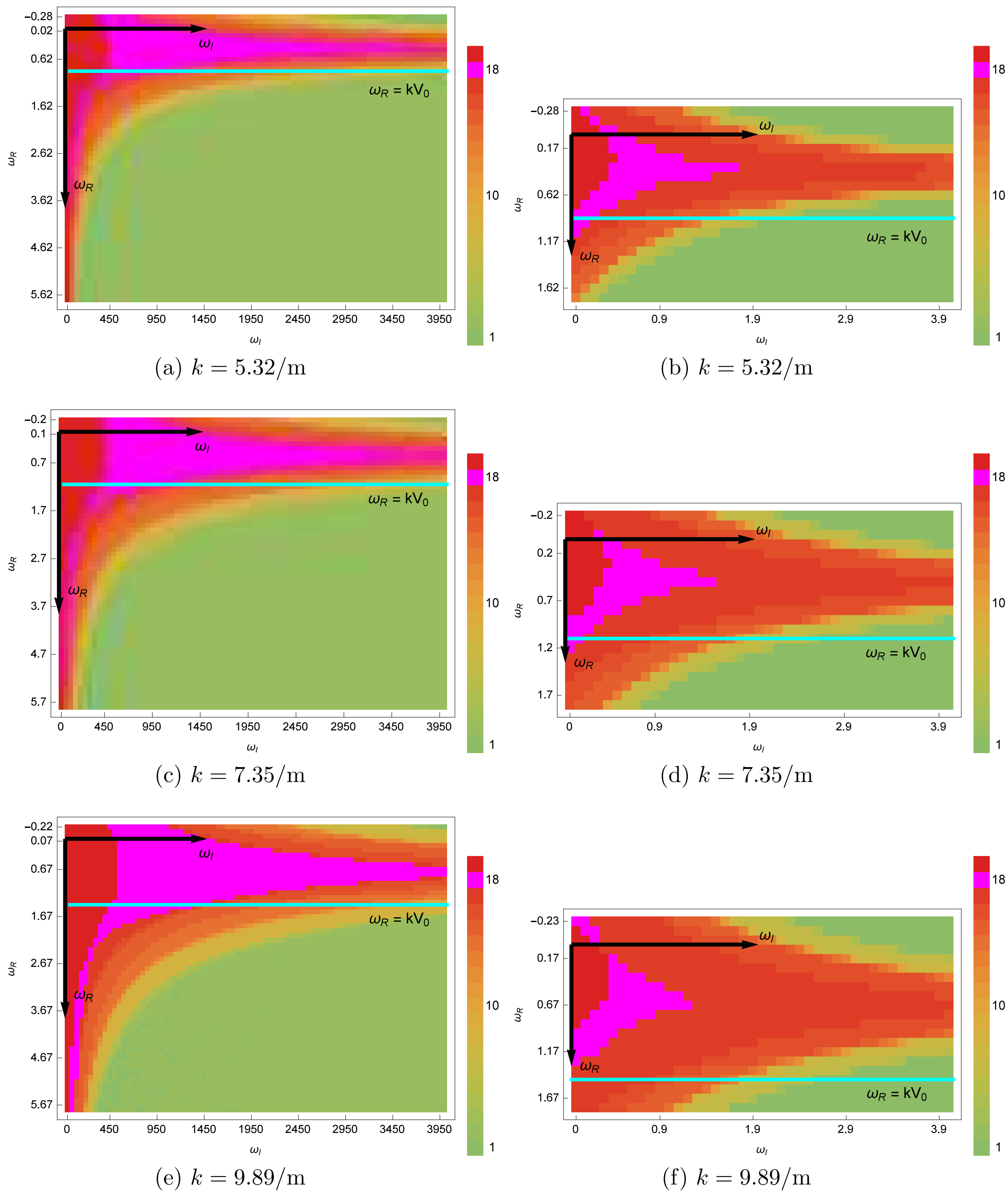
[3] Jeffreys, H. On the formation of water waves by wind. *Proceedings of the Royal Society of London. Series A 107*, 742 (1925), 189–206.

[4] Jeffreys, H. On the formation of water waves by wind (second paper). *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character 110*, 754 (1926), 241–247.

[5] Lamb, H. *Hydrodynamics*, 6th ed. Cambridge University Press, 1932.

[6] Lyons, D. W., Walck, S. N., and Blanda, S. A. Classification of nonproduct states with maximum stabilizer dimension. *Physical Review A 77*, 2 (2008), 022309.

[7] Miles, J. On the generation of surface waves by shear flows. *Journal of Fluid Mechanics 3*, 2 (1957a), 185–204.



(a) $k = 5.32$/m (b) $k = 5.32$/m

(c) $k = 7.35$/m (d) $k = 7.35$/m

(e) $k = 9.89$/m (f) $k = 9.89$/m

Figure 2: Contours of rank from matrix M for three different wavenumbers, k, for $U^{\infty} = 3.5$m/s and $V_0 = 0.149$m/s. The first column uses $z_1 = 2.49 \times 10^{-4}$m; the second uses $z_1 = 0.01$m. The `rank` $M = 18$ case is magenta. The cyan line in each plot denotes $\omega_R = kV_0$.