

Homework Assignment #9

MATH 251: Ordinary and Partial Differential Equations

Due Friday, November 9, 2012

In order to get full credit on any of the questions, you are required to
show your work!

1 - 4 Determine the type and stability of the critical point at $(0,0)$ of each system below.

1.

$$\mathbf{x}' = \begin{bmatrix} 2 & 7 \\ -5 & -10 \end{bmatrix} \mathbf{x}.$$

$$\det(A - rI) = \det \begin{bmatrix} 2-r & 7 \\ -5 & -10-r \end{bmatrix} = (2-r)(-10-r) - (7)(-5) = 0$$
$$r^2 + 8r + 15 = 0 \Rightarrow (r+3)(r+5) = 0$$

Eigenvalues $r = -3, -5$

\Rightarrow Node, asymptotically stable

2.

$$\mathbf{x}' = \begin{bmatrix} -3 & 6 \\ -3 & 3 \end{bmatrix} \mathbf{x}.$$

$$\det(A - rI) = \det \begin{bmatrix} -3-r & 6 \\ -3 & 3-r \end{bmatrix} = (-3-r)(3-r) - (6)(-3) = 0$$
$$r^2 + 9 = 0 \Rightarrow r = \pm \sqrt{-9} \Rightarrow r = \pm 3i$$

\Rightarrow Center, stable

3.

$$\mathbf{x}' = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \mathbf{x}.$$

$$\det(A - rI) = \det \begin{bmatrix} -1-r & -1 \\ 1 & -1-r \end{bmatrix} = (-1-r)(-1-r) - (-1)(1) = 0$$

$$r^2 + 2r + 2 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2} = -1 \pm i$$

\Rightarrow Spiral point, asymptotically stable

4.

$$\mathbf{x}' = \begin{bmatrix} 6 & -5 \\ 0 & 6 \end{bmatrix} \mathbf{x}.$$

Triangular matrix \Rightarrow eigenvalues on diagonal

$\Rightarrow r = 6$ (repeated) (Note: will only have one eigenvector)

\Rightarrow Improper Node, unstable

5. (i) For what value(s) of b will the system below have an improper node at $(0,0)$? (ii) For what value(s) of b will the system below have a spiral point at $(0,0)$?

$$\mathbf{x}' = \begin{bmatrix} 5 & b \\ 2 & -1 \end{bmatrix} \mathbf{x}$$

$$\det(A - rI) = \det \begin{bmatrix} 5-r & b \\ 2 & -1-r \end{bmatrix} = (5-r)(-1-r) - 2b = 0$$

$$r^2 - 4r - 5 - 2b = 0.$$

(i) Improper node at $(0,0)$ if we have repeated real eigenvalue with only one eigenvector.

$$(r-2)^2 = 0 \Rightarrow r=2 \text{ (repeated)}$$

$$\Rightarrow 4 = -5 - 2b \Rightarrow 9 = -2b \Rightarrow b = -\frac{9}{2}$$

(ii) Spiral point at $(0,0)$ if we have complex eigenvalues with nonzero real part

$$r = \frac{+4 \pm \sqrt{16-4(-5-2b)}}{2} \quad \text{Need } 16 - 4(1)(-5-2b) < 0$$

$$-4(-5-2b) < -16 \Rightarrow -5-2b > 4 \Rightarrow -2b > 9$$

$$\Rightarrow b < -\frac{9}{2}$$

6 - 7 Find the critical point of each nonhomogeneous linear system given.
Then determine the type and stability of the critical point.

6.

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 6 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -6 \\ 24 \end{bmatrix}.$$

$$\begin{aligned} 0 &= x_1 + 2x_2 - 6 \Rightarrow 6 = x_1 + 2x_2 \\ 0 &= 6x_1 - 3x_2 + 24 \Rightarrow -24 = 6x_1 - 3x_2 \end{aligned} \Rightarrow \boxed{\text{Critical Point: } (-2, 4)}$$

$$\det(A - rI) = \det \begin{bmatrix} 1-r & 2 \\ 6 & -3-r \end{bmatrix} = (1-r)(-3-r) - (2)(6) = 0$$

$$r^2 + 2r - 15 = 0 \Rightarrow (r+5)(r-3) = 0$$

Eigenvalues $r = -5, 3$

\Rightarrow Saddle Point, Unstable

7.

$$\mathbf{x}' = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 12 \\ -3 \end{bmatrix}.$$

$$\begin{aligned} 0 &= 3x_1 + 12 \Rightarrow -12 = 3x_1 \\ 0 &= 3x_2 - 3 \Rightarrow 3 = 3x_2 \end{aligned} \Rightarrow \boxed{\text{Critical Point: } (-4, 1)}$$

$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ Diagonal matrix has eigenvalues on the diagonal $\Rightarrow r = 3$ (repeated)

$A = 3I \Rightarrow$ two eigenvectors

\Rightarrow Proper Node, Unstable

(or Star Point)

8 - 9 Find all the critical point(s) of each nonlinear system given. Then determine the type and stability of each critical point.
8.

$$\begin{array}{l} \text{Critical Points: } \\ \begin{aligned} 0 &= (x+3)y & x' &= xy + 3y \\ 0 &= (y-3)x & y' &= xy - 3x \end{aligned} \Rightarrow \begin{array}{l} x = -3 \text{ or } y = 0 \\ y = 3 \text{ or } x = 0 \end{array} \Rightarrow \boxed{\text{Critical Points: } (0,0) \text{ and } (-3,3)} \end{array}$$

$$J = \begin{bmatrix} y & x+3 \\ y-3 & x \end{bmatrix}$$

For $(0,0)$:

$$A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \quad \text{Characteristic Equation: } r^2 + 9 = 0$$

$$\Rightarrow r = \pm 3i$$

$\Rightarrow (0,0)$ is a stable center

For $(-3,3)$:

$$A = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} \Rightarrow \text{Eigenvalues } r = 3, -3$$

$\Rightarrow (-3,3)$ is an unstable saddle point

9.

Critical Points:

$$\begin{aligned}x' &= x^2y + 3xy - 10y \\y' &= xy - 4x\end{aligned}$$

$$\begin{aligned}0 &= (x^2 + 3x - 10)y = (x+5)(x-2)y \Rightarrow x = -5, x = 2, \text{ or } y = 0 \\0 &= (y-4)x \Rightarrow y = 4 \text{ or } x = 0\end{aligned}$$

Critical Points: $(0,0)$, $(-5,4)$, and $(2,4)$

$$J = \begin{bmatrix} 2xy + 3y & x^2 + 3x - 10 \\ y-4 & x \end{bmatrix}$$

For $(0,0)$:

$$A = \begin{bmatrix} 0 & -10 \\ -4 & 0 \end{bmatrix} \quad \begin{array}{l} \text{Characteristic} \\ \text{Equation: } r^2 - 40 = 0 \Rightarrow r = \pm 2\sqrt{10} \end{array}$$

 $(0,0)$ is an unstable saddle pointFor $(2,4)$:

$$A = \begin{bmatrix} 28 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow r = 28, 2$$

 $(2,4)$ is an unstable nodeFor $(-5,4)$:

$$A = \begin{bmatrix} -28 & 0 \\ 0 & -5 \end{bmatrix} \Rightarrow r = -28, -5$$

 $(-5,4)$ is an asymptotically stable node