Lecture Notes for Math 251
Lecture 26: Section 9.1 and 9.2 - Phase Portraits of 2×2 Linear Systems; Autonomous Systems and Stability

1 Phase Portraits of Linear Systems

Consider a system of linear differential equations $\mathbf{x}' = A\mathbf{x}$. Its **phase portrait** is a representative set of its solutions, plotted as parametric curves (with t as the parameter) on the Cartesian plane tracing the path of each particular solution $(x, y) = (x_1(t), x_2(t))$, $-\infty < t < \infty$. Similar to a direction field, a phase portrait is a graphical tool to visualize how the solutions of a given system of differential equations would behave in the long run.

Definition 1. In this context, the Cartesian plane where the phase portrait resides is called the **phase plane**. The parametric curves traced by the solutions are sometimes also called their **trajectories**.

Remark: It is quite labor-intensive, but it is possible to sketch the phase portrait by hand without first having to solve the system of equations that it represents. Just like a direction field, a phase portrait can be a tool to predict the behaviors of a system's solutions. To do so, we draw a grid on the phase plane. Then, at each grid point $\mathbf{x} = (\alpha, \beta)$, we can calculate the solution trajectory's instantaneous direction of motion at that point by using the given system of equations to compute the tangent/velocity vector $\mathbf{x}'$. Namely, plug in $\mathbf{x} = (\alpha, \beta)$ to compute $\mathbf{x}' = A\mathbf{x}$.

We will examine the phase portrait of a linear system of differential equations. We will classify the type and stability of the equilibrium solution of a given linear system by the shape and behavior of its phase portrait.

2 Equilibrium Solution (aka Critical Point or Stationary Point)

An equilibrium solution of the system $\mathbf{x}' = A\mathbf{x}$ is a point (x_1, x_2) where $\mathbf{x}' = \mathbf{0}$, that is, where $x_1' = 0 = x_2'$. An equilibrium solution is a constant solution of the system and is usually called a **critical point**. For a linear system $\mathbf{x}' = A\mathbf{x}$, an equilibrium solution occurs at each solution of the system (of homogeneous algebraic equations) $A\mathbf{x} = \mathbf{0}$. As we have seen, such a system has exactly one solution, located at the origin, if $\det(A) \neq 0$. If $\det(A) = 0$, then there are infinitely many solutions.

For our purpose, and unless otherwise noted, we will only consider systems of linear differential equations whose coefficient matrix A has nonzero determinant. That is, we will only consider systems where the origin is the only critical point.

Note: A matrix could only have zero as one of its eigenvalues if and only if its determinant is also zero. Therefore, since we limit ourselves to consider only those systems where $\det(A) \neq 0$, we will not encounter any matrix with zero as an eigenvalue.

3 Classification of Critical Points

Similar to the earlier discussion on the equilibrium solutions of a single first order differential equation using the direction field, we will presently classify the critical points of various systems of first order linear differential equations by their **stability**. In addition, due to the truly two-dimensional nature of the parametric curves, we will also classify the **type** of those critical points by their shapes (or, rather, by the shape formed by the trajectories about each critical point).

Comment: The accurate tracing of the parametric curves of the solutions is not an easy task without electronic aids. However, we can obtain very reasonable approximations of a trajectory by using the same idea behind the direction field, namely the tangent line approximation. At each point $\mathbf{x} = (x_1, x_2)$ on the ty-plane, the direction of motion of the solution curve that passes through the point is determined by the direction vector (i.e. the **tangent vector**) $\mathbf{x}'$, the derivative of the solution vector $\mathbf{x}$, evaluated at the given point. The tangent vector at each given point can be calculated directly from the given matrix-vector equation $\mathbf{x}' = A\mathbf{x}$, using the position vector $\mathbf{x} = (x_1, x_2)$. Like working with a direction field, there is no need to find the solution first before performing this approximation.

4 Autonomous Systems and Stability

Given: $\mathbf{x}' = A\mathbf{x}$, where there is only one critical point, at $(0, 0)$.

4.1 Case 1: Distinct Real Eigenvalues

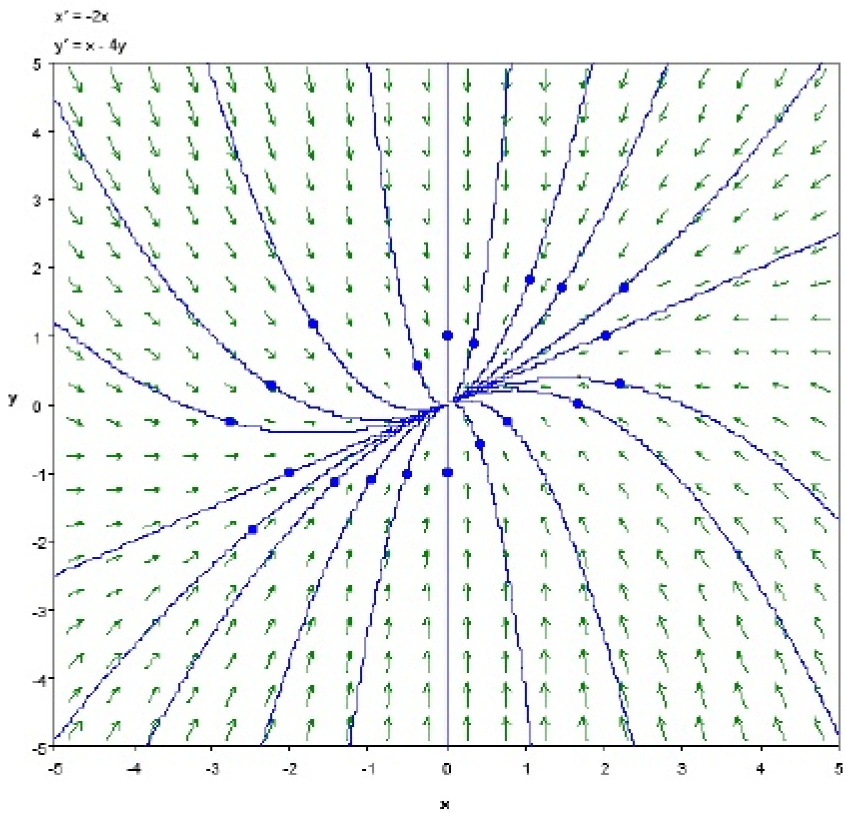
The general solution is

$$\mathbf{x} = C_1\mathbf{k_1}e^{r_1 t} + C_2\mathbf{k_2}e^{r_2 t}.$$

1. When r_1 and r_2 are both positive or are both negative

The phase portrait shows trajectories either moving away from the critical point to infinite-distance away (when $r > 0$), or moving directly towards, and converting to, the critical point(when $r < 0$). The trajectories that are the eigenvectors move in straight lines. The rest of the trajectories move, initially when near the critical point, roughly in the same direction as the eigenvector of the eigenvalue with the smaller absolute value. Then, farther away, they would bend toward the direction of the eigenvector of the eigenvalue with the larger absolute value. The trajectories either move away from the critical point to infinite-distance away (when r are both positive), or eventually converge to the critical point (when r are both negative). This type of critical point is called a **node**. It is **asymptotically stable** if r are both negative, **unstable** if r are both positive.

Two distinct real eigenvalues, both of the same sign



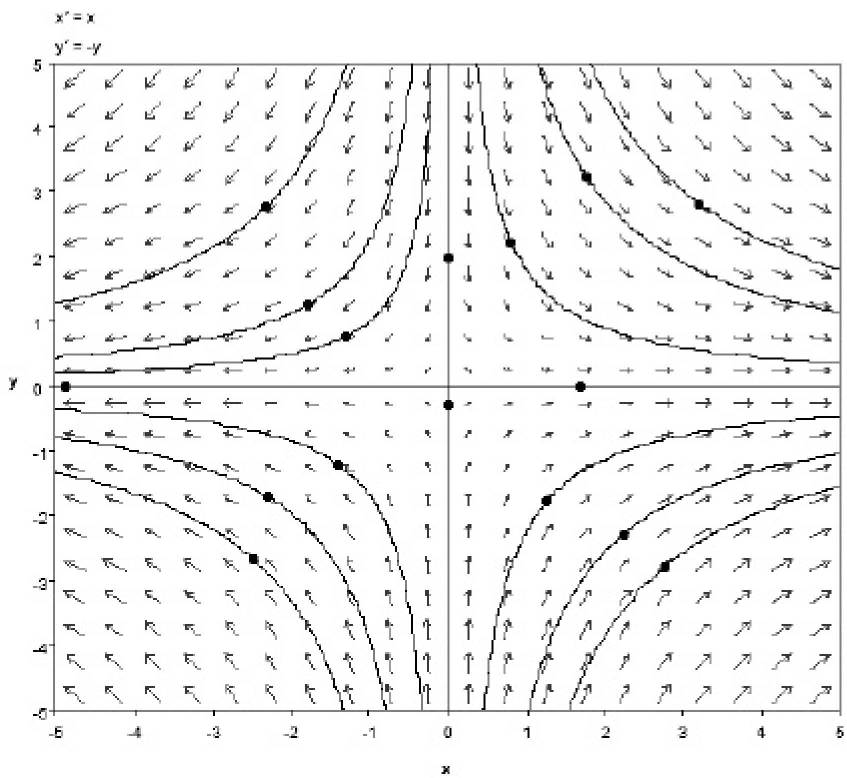
Type: Node

Stability: It is **unstable** if both eigenvalues are positive; **asymptotically stable** if they are both negative.

2. When r_1 and r_2 have opposite signs (say $r_1 > 0$ and $r_2 < 0$)

In this type of phase portrait, the trajectories given by the eigenvectors of the negative eigenvalue initially start at infinite-distance away, then move toward and eventually converge at the critical point. The trajectories that represent the eigenvectors of the positive eigenvalue move in exactly the opposite way: they start at the critical point, then diverge to infinite-distance out. Every other trajectory starts at infinite-distance away. All the while, it would roughly follow the two sets of eigenvectors. This type of critical point is called a **saddle point**. It is always **unstable**.

Two distinct real eigenvalues, opposite signs



Type: Saddle Point
Stability: It is always **unstable**.

4.2 Case 2: Repeated Real Eigenvalue

3. When there are two linearly independent eigenvectors $\mathbf{k_1}$ and $\mathbf{k_2}$.

The general solution is

$$\mathbf{x} = C_1\mathbf{k_1}e^{rt} + C_2\mathbf{k_2}e^{rt} = e^{rt}(C_1\mathbf{k_1} + C_2\mathbf{k_2}).$$

Every nonzero solution traces a straight-line trajectory, in the direction given by the vector $C_1\mathbf{k_1} + C_2\mathbf{k_2}$. The phase portrait thus has a distinct star-burst shape. The trajectories either move directly away from the critical point to infinite-distance away (when $r > 0$), or more directly toward (and converge to) the critical point (when $r < 0$). This type of critical point is called a **proper node** (or a **star point**). It is **asymptotically stable** if $r < 0$ and **unstable** if $r > 0$.

Note: For 2×2 systems of linear differential equations, this will occur if, and only if, the coefficient matrix A is a constant multiple of the identity matrix:

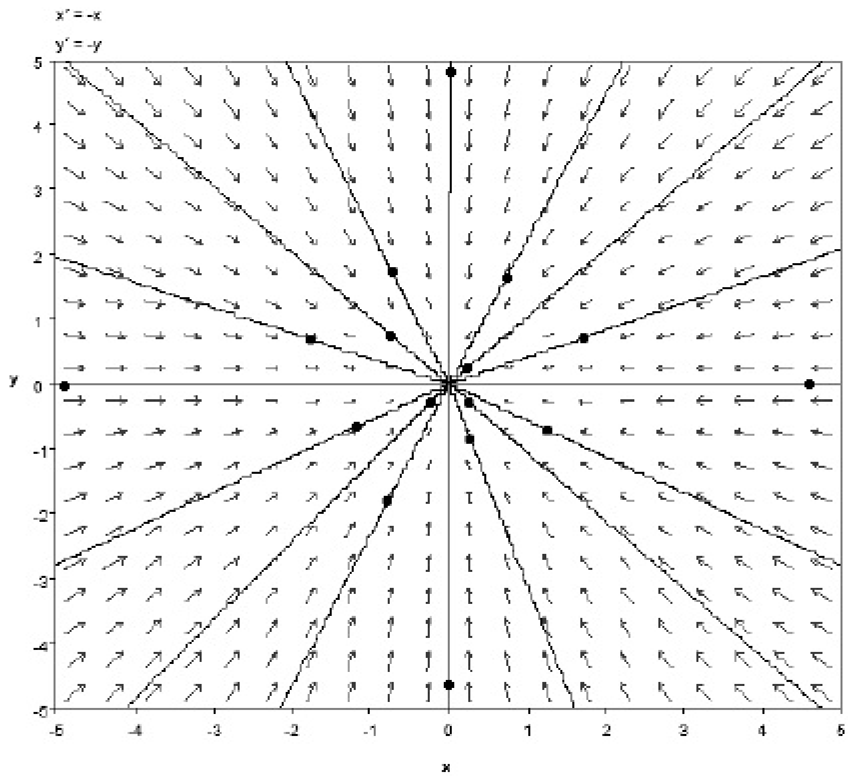
$$A = \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \quad \text{for any constant } \alpha.$$

In the case of $\alpha = 0$, the solution is

$$\mathbf{x} = C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

Every solution, in this case, is an equilibrium solution. Therefore, every trajectory on its phase portrait consists of a single point, and every point on the phase plane is a trajectory.

A repeated real eigenvalue, two linearly independent eigenvectors



Type: Proper Node (or Star Point)

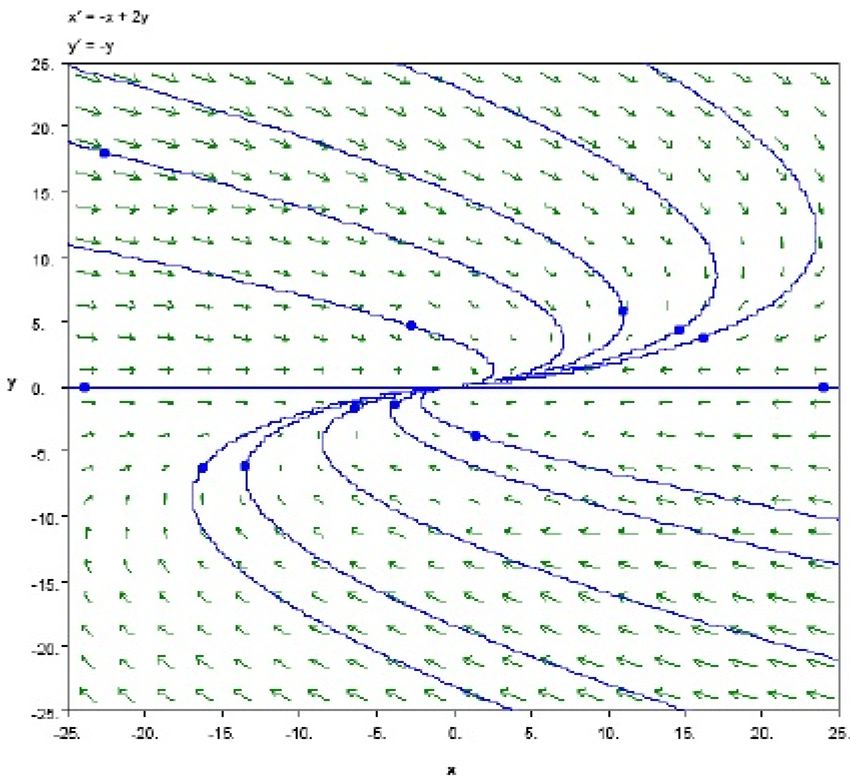
Stability: It is **unstable** if the eigenvalue is positive; **asymptotically stable** if the eigenvalue is negative.

4. When there is only one linearly independent eigenvector $\mathbf{k}$.
Then, the general solution is

$$\mathbf{x} = C_1 \mathbf{k} e^{rt} + C_2(\mathbf{k} t e^{rt} + \eta e^{rt}).$$

The phase portrait shares characteristics with that of a node. With only one eigenvector, it is a degenerated-looking node that is a cross between a node and a spiral point (see #6 below). The trajectories either all diverge away from the critical point to infinite-distance away (when $r > 0$), or all converge to the critical point (when $r < 0$). This type of critical point is called an **improper node**. It is **asymptotically stable** if $r < 0$, **unstable** if $r > 0$.

A repeated real eigenvalue, only one linearly independent eigenvector



Type: Improper Node

Stability: It is **unstable** if the eigenvalue is positive; **asymptotically stable** if the eigenvalue is negative.

5 Case 3: Complex Conjugate Eigenvalues

The general solution is

$$\mathbf{x} = C_1 e^{\lambda t}(\mathbf{a}\cos(\mu t) - \mathbf{b}\sin(\mu t)) + C_2 e^{\lambda t}(\mathbf{a}\sin(\mu t) + \mathbf{b}\cos(\mu t)).$$

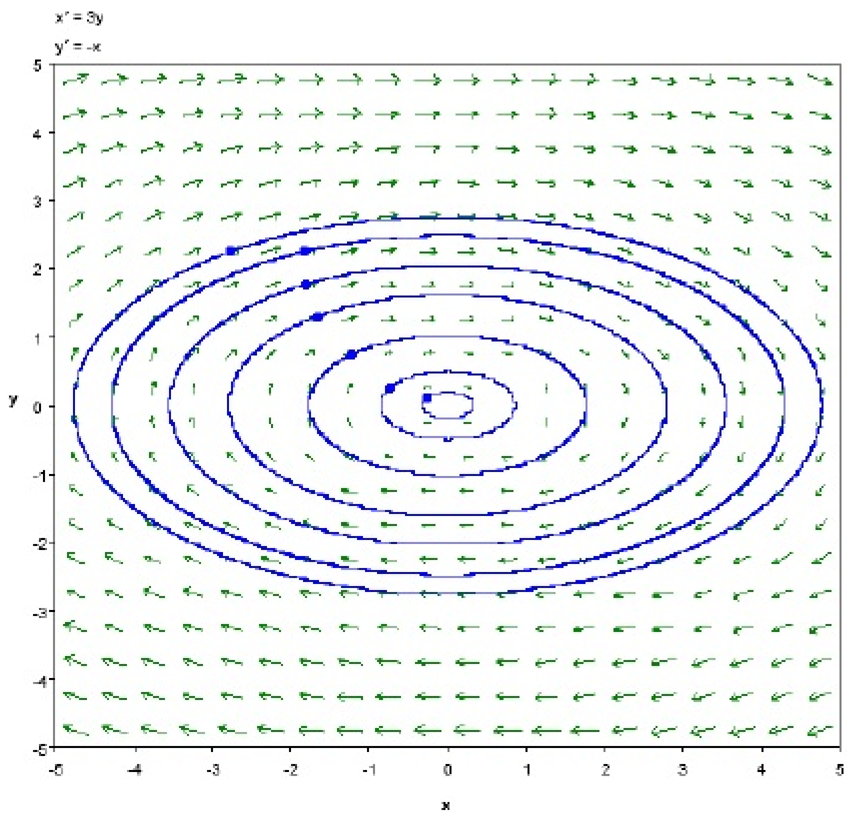
5. When the real part of λ is zero.

In this case, the trajectories neither converge to the critical point nor move to infinite-distance away. Rather, they stay in constant, elliptical (or, rarely, circular) orbits. This type of critical point is called a **center**. It has a unique stability classification shared by no other: **stable** (or **neutrally stable**). It is **NOT** asymptotically stable and one should not confuse them.

6. When the real part of λ is nonzero.

The trajectories still retain the elliptical traces as in the previous case. However, with each revolution, their distances from the critical point grow/decay exponentially according to the term $e^{\lambda t}$. Therefore, the phase portrait shows trajectories that spiral away from the critical point to infinite-distance away (when $\lambda > 0$), or trajectories that spiral toward (and converge to) the critical point (when $\lambda < 0$). This type of critical point is called a **spiral point**. It is **asymptotically stable** if $\lambda < 0$, it is **unstable** if $\lambda > 0$.

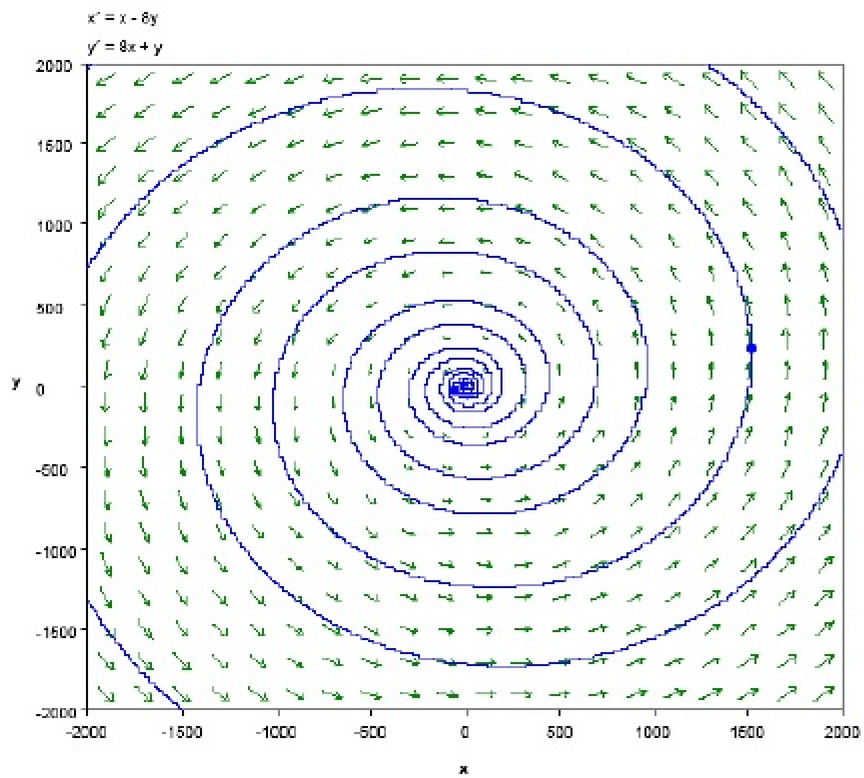
Complex eigenvalues, with real part zero (purely imaginary numbers)



Type: Center

Stability: **Stable** (but NOT asymptotically stable); sometimes it is referred to as **neutrally stable**.

Complex eigenvalues, with nonzero real part



Type: Spiral Point

Stability: It is **unstable** if the eigenvalues have positive real part; **asymptotically stable** if the eigenvalues have negative real part.

6 Summary of Stability Classification

Asymptotically Stable - All trajectories of its solutions converge to the critical point as $t \to \infty$. A critical point is asymptotically stable if all of A's eigenvalues are negative or have negative real part for complex eigenvalues.

Unstable - All trajectories (or all but a few, in the case of a saddle point) start out at the critical point at $t \to -\infty$, then move away to infinitely distant out as $t \to \infty$. A critical point is unstable if at least one of A's eigenvalues is positive or has positive real part for complex eigenvalues.

Stable (or **Neutrally Stable**) - Each trajectory moves about the critical point within a finite range of distances. It never moves out to infinitely distant, nor (unlike the case of asymptotically stable) does it ever go to the critical point. A critical point is stable if A's eigenvalues are purely imaginary.

In short, as t increases, if all (or almost all) trajectories:

1. converge to the critical point $\to$ **asymptotically stable**
2. move away from the critical point to infinitely far away $\to$ **unstable**
3. stay in a fixed orbit within a finite (i.e., bounded) range of distances away from the critical point $\to$ **stable** (or **neutrally stable**)

7 Nonhomogeneous Linear Systems with Constant Coefficients

Now let us consider the nonhomogeneous system

$$\mathbf{x}' = A\mathbf{x} + \mathbf{b},$$

where $\mathbf{b}$ is a constant vector. The system above is explicitly:

$$x_1' = ax_1 + bx_2 + g_1$$

$$x_2' = cx_1 + dx_2 + g_2$$

As before, we can find the critical point by setting $x_1' = x_2' = 0$ and solve the resulting nonhomogeneous system of algebraic equations. The origin will no longer be a critical point, since the zero vector is **never** a solution of a nonhomogeneous linear system. Instead, the unique critical point (as long as A has nonzero determinant, there remains exactly one critical point) will be located at the solution of the system of algebraic equations:

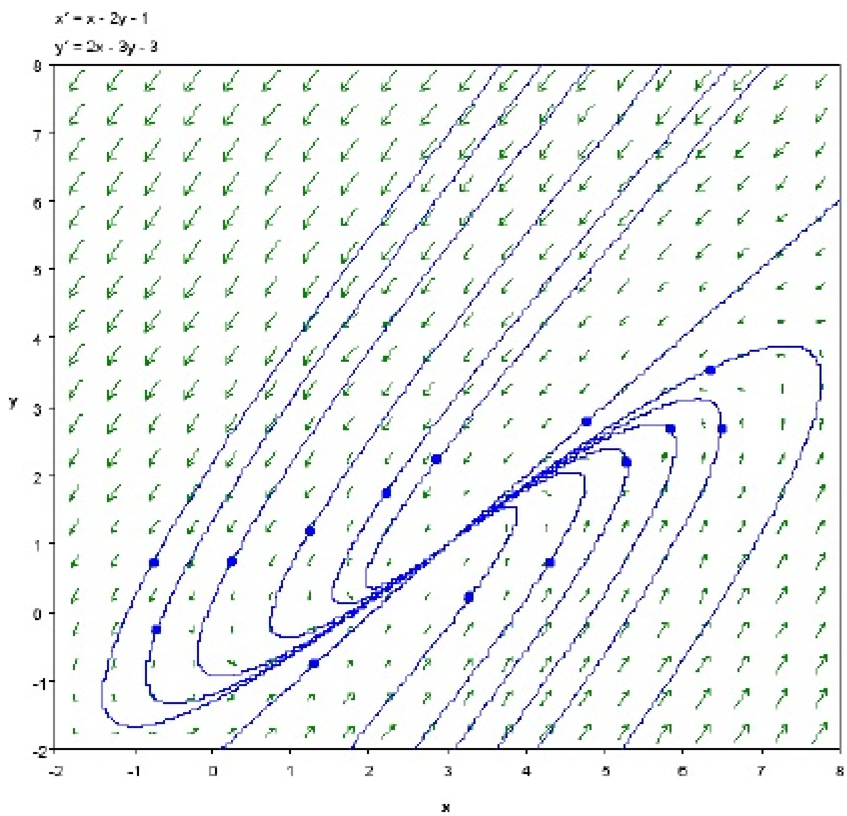
$$0 = ax_1 + bx_2 + g_1$$

$$0 = cx_1 + dx_2 + g_2$$

Once we have found the critical point, say it is the point $(x_1, x_2) = (\alpha, \beta)$, it then could be moved to $(0, 0)$ via the translations $\chi_1 = x_1 - \alpha$ and $\chi_2 = x_2 - \beta$. The result after the translation would be the homogeneous linear system $\chi' = A\chi$. The two systems (before and after the translations) have the same coefficient matrix. Their respective critical points will also have identical type and stability classification. Therefore, to determine the type and stability of the critical point of the given nonhomogeneous system, all we need to do is to disregard $\mathbf{b}$, then take its coefficient matrix A and use its eigenvalues for the determination, in exactly the same way as we would do with the corresponding homogeneous system of equations.

Example 1.

$$x_1' = x_1 - 2x_2 - 1$$
$$x_2' = 2x_1 - 3x_2 - 3$$



Example 2.

$$x_1' = -2x_1 - 6x_2 + 8$$
$$x_2' = 8x_1 + 4x_2 - 12$$