Math 170-01 Calculus I Final Exam Review Solutions

1. Find the following limits:

   (a) $\lim\limits_{x\to -1} \dfrac{x^3-x^2}{x-1} = 1$

   (b) $\lim\limits_{x\to 2^-} \dfrac{x+2}{x-2} = -\infty$

   (c) $\lim\limits_{x\to \infty} \dfrac{2x^2-6}{x^2+5x} = 2$

   (d) $\lim\limits_{x\to \infty} \dfrac{x^3-x^2+10}{3x^2-4x} = \infty$

   (e) $\lim\limits_{x\to 0} \dfrac{x\sin(x)}{1-\cos(x)} = 2$

   (f) $\lim\limits_{x\to 0} \ln(\sin(2x)) - \ln(\tan(x)) = \ln(2)$

   (g) $\lim\limits_{x\to 0^+} \dfrac{\cot(x)}{\ln(x)} = -\infty$

   (h) $\lim\limits_{x\to \infty} x\sin(\pi/x) = \pi$

2. Find any values of x (if they exist) where the function $f(x)$ is not continuous:

   (a) $f(x) = \dfrac{4x+1}{x^2-1}$, $x = \pm 1$

   (b) $f(x) = |x^2+3|$, cont everywhere

   (c) $f(x) = \dfrac{x+5}{|x^2+5x|}$, $x = 0, -5$

   (d) $f(x) = e^{\ln(x)}$, $x < 0$

3. Use the limit definition of the derivative to find the slope of the tangent line to the graph of $f(x) = x^2+1$ at a general x value. Then, use it to find the slope of the tangent line to the graph of f at $x = 4$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2+1-x^2-1}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} 2x+h = 2x$$

$$f'(4) = 2(4) = 8$$

4. Find $f'(x)$ for each of the following:

   (a) $f(x) = \sin^3(x) + 4\cos(x) = 3\sin^2(x)\cos(x) - 4\sin(x)$

   (b) $f(x) = \sqrt{5x-2}(x+3)^2 = 2\sqrt{5x-2}(x+3) + \frac{5(x+3)^2}{2\sqrt{5x-2}}$

   (c) $f(x) = \left(\dfrac{4x-2}{x^3}\right)^2 = 2\left(\frac{4x-2}{x^3}\right)\left(\frac{-8x^3+6x^2}{x^6}\right)$

   (d) $f(x) = \tan^3(x^4) = 12x^3\sec^4(x^4)$

   (e) $f(x) = \sqrt[3]{5+\sqrt{x}} = \frac{1}{6}(5+\sqrt{x})^{-2/3}x^{-1/2}$

   (f) $f(x) = 2xe^{\sqrt{x}} = 2e^{\sqrt{x}} + 2x\frac{1}{\sqrt{x}}e^{\sqrt{x}}$

   (g) $f(x) = \cos^3\left(\dfrac{x}{x+1}\right) = -3\left(\frac{1}{(x+1)^2}\right)\cos\left(\frac{x}{x+1}\right)^2\sin\left(\frac{x}{x+1}\right)$

   (h) $f(x) = \dfrac{6}{1+3e^x} = -\frac{18e^x}{(3e^x+1)^2}$

(i) $f(x) = \cos^{-1} x^2 = -\frac{2x}{\sqrt{1-x^4}}$

(j) $f(x) = \sqrt[3]{\ln(x) + 1} = \frac{1}{3x}(\ln(x) + 1)^{-2/3}$

5. Find the equation of the tangent line to the graph of $y = \ln(2 - x^2)$ at $x = 2$.
2 is not in the domain of the function, so the tangent line does not exist there. But, at values of x in the domain of the function, we'd find the tangent line by finding $y' = \frac{-2x}{2-x^2}$, finding the slope m by evaluating y' at x, finding another point on the line by finding the point $(x, y(x))$, then writing the point slope form of the line.

6. Find y' when

(a) $x^3 + xy + y^3 = x, \quad y' = \frac{1-y-3x^2}{x+3y^2}$

(b) $3xy^2 - 6x + 3y^3 = 9, \quad y' = \frac{6-3y^2}{9y^2+6xy}$

7. Find the derivative of $y = \dfrac{x^3}{\sqrt{x^2+3}}$ using logarithmic differentiation.

$$y' = \left(\frac{3}{x} - \frac{x}{x^2+3}\right)\left(\frac{x^3}{\sqrt{x^2+3}}\right)$$

8. Use an appropriate local linearization to approximate $(1.97)^3$.
Let $f(x) = x^3$, $x_0 = 2$.

$$(1.97)^3 \approx f(2) + f'(2)(1.97 - 2) = 8 + 3(2)^2(-\frac{3}{100}) = 8 - \frac{36}{100}$$

9. A rocket, rising vertically, is tracked by a radar station that is on the ground 5 miles from the launchpad. How fast is the rocket rising when it is 4 miles high and its distance from the radar station is increasing at a rate of 2000 miles per hour?
$500\sqrt{41}$ miles per hour

10. For each of the following functions, find (a) the intervals on which the function is increasing or decreasing and (b) the intervals on which the function is concave up or concave down.

(a) $f(x) = \dfrac{x-2}{(x^2-x+1)^2}$

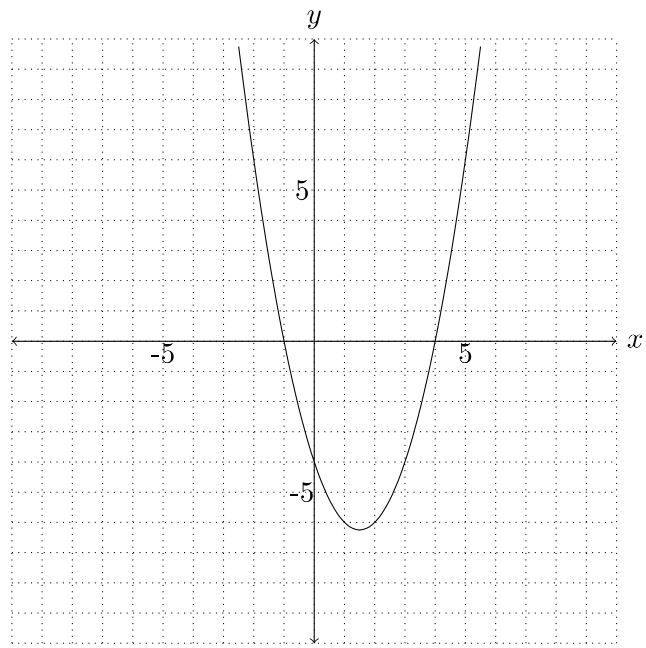
Decreasing on $(-\infty, \frac{3-\sqrt{5}}{2})$, $(\frac{3+\sqrt{5}}{2}, \infty)$
Increasing on $(\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2})$
Concave down on $(-\infty, 0)$, $(\frac{4-\sqrt{6}}{2}, \frac{4+\sqrt{6}}{2})$
Concave up on $(0, \frac{4-\sqrt{6}}{2})$, $(\frac{4+\sqrt{6}}{2}, \infty)$

(b) $f(x) = x^3 \ln(x)$

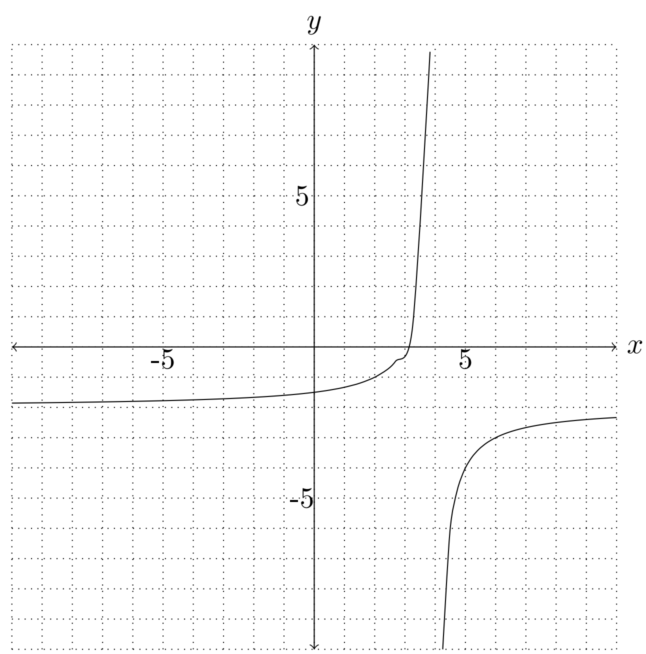
Decreasing on $(-\infty, \frac{1}{\sqrt[3]{e}})$
Increasing on $(\frac{1}{\sqrt[3]{e}}, \infty)$
Concave down on $(-\infty, \frac{1}{\sqrt[6]{e^5}})$
Concave up on $(\frac{1}{\sqrt[6]{e^5}}, \infty)$

11. Sketch a graph of the following functions by calculating critical values, inflection points, intercepts, intervals of increasing and decreasing, intervals of concavity, asymptotes, etc.

(a) $f(x) = x^2 - 3x - 4$



(b) $f(x) = \dfrac{2x-6}{4-x}$



12. Sketch the graph of a function $f(x)$ where

$$f(0) = -1, \qquad f'(0) = 0, \qquad f'(-1) < 0, \qquad \lim_{x\to\infty} f(x) = \infty, \qquad \lim_{x\to-\infty} f(x) = 3$$

13. A rectangular area of 3200 square feet is to be fenced off. Two opposite sides will use fencing costing \$1 per foot, and the remaining sides will use fencing costing \$2 per foot. Find the dimensions of the rectangle with the lowest possible cost.
If y is the length of a side that costs \$2 per foot, and x is the length of a side costing \$1 per foot, then the dimensions with the lowest possible cost would be $x = 800$ ft and $y = 400$ ft. The restriction equation would be $3200 = xy$ and the cost equation (that we need to find a minimum of) would be $C = 2x + 4y$.

14. Verify that the hypotheses of Rolle's Theorem are satisfied on the interval $[-1, 3]$ for the function $f(x) = \ln(4 + 2x - x^2)$, and find all values of c in that interval that satisfy the conclusion of the theorem.
f is continuous on the interval $[-1, 3]$, and its derivative $f'(x) = \frac{2-2x}{4+2x-x^2}$ is defined on $(-1, 3)$. Also, $f(-1) = f(3) = 0$, so all hypothesis are met.

$$0 = f'(c) = \frac{2 - 2c}{4 + 2c - c^2} \qquad \Rightarrow \qquad c = 1$$

15. Find the integrals:

(a) $\int (4x^3 - 6x + 8)\, dx = x^4 - 3x^2 + 8x + C$

(b) $\int_1^4 \frac{4}{x^2}\, dx = \frac{-4}{x}\Big]_1^4 = 3$

(c) $\int_1^{\sqrt{2}} xe^{-x^2}\, dx = \frac{1}{2}\int_{-2}^{-1} e^u\, du = \frac{1}{2}(e^{-1} - e^{-2})$

(d) $\int_{-1}^1 \frac{dx}{1 + x^2} = \tan(x)\Big]_{-1}^1 = \pi/2$

(e) $\int \tan(2\theta)\, d\theta = -\frac{1}{2}\ln|\cos(2\theta)| + C$

(f) $\int_{1/2}^1 \frac{1}{2x}\, dx = \frac{1}{2}\ln(x)\Big]_{1/2}^1 = -\ln(1/2)$

(g) $\int_{-2}^{-1} \frac{x}{(x^2 + 2)^3}\, dx = -\frac{1}{48}$

(h) $\int_0^2 |2x - 3|\, dx = \int_0^{3/2} 3 - 2x\, dx + \int_{3/2}^2 2x - 3\, dx = 5/2$

(i) $\int x(1 + x^3)\, dx = \frac{1}{2}x^2 + \frac{1}{5}x^5 + C$

16. Let $F(x) = \int_1^x (t^3 + 1)\, dt$. Use the Fundamental Theorem of Calculus to find $F'(x)$.

$$F'(x) = x^3 + 1$$