Exam III Review Solutions
Math 170 Calculus I

1. Sketch a graph of a function $f(x)$ that satisfies $f(-2) = -1$, $f(0) = 0$, $f(2) = 1$, $\lim_{x \to -\infty} f(x) = 0$, and $\lim_{x \to \infty} f(x) = 0$.

There are lots of possible graphs here. We just need a curve that goes through the three points $(-2, -1)$, $(0, 0)$, and $(2, 1)$ with horizontal asymptote $y = 0$.

2. Find the absolute maximum and minimum of $f(x) = 15x^4 - 15x^2 + 31$ on $[-1, 2]$.

Find the critical values:

$$f'(x) = 60x^3 - 30x = 30x(2x^2 - 1) = 0$$

when $x = 0$ and $x = \pm\frac{1}{\sqrt{2}}$. All of these are in the interval, so the max and min could occur at either of these values or at an endpoint:

$$f(0) = 15(0)^4 - 15(0)^2 + 31 = 31$$

$$f(1/\sqrt{2}) = 15(1/\sqrt{2})^4 - 15(1/\sqrt{2})^2 + 31 = 109/4$$
$$f(-1/\sqrt{2}) = 15(-1/\sqrt{2})^4 - 15(-1/\sqrt{2})^2 + 31 = 109/4$$
$$f(-1) = 15(-1)^4 - 15(-1)^2 + 31 = 31$$
$$f(2) = 15(2)^4 - 15(2)^2 + 31 = 211$$

The absolute minimum of the function is $109/4$, which occurs at $x = -1/\sqrt{2}$ and $x = 1/\sqrt{2}$, and the absolute maximum is 211, which occurs at the endpoint $x = 2$.

3. For the function $f(x) = x^2 - 3x - 4$:

   (a) Find any x and y intercepts of f.

   Since
   $$f(0) = (0)^2 - 3(0) - 4 = -4$$
   then the y-intercept is $(0, -4)$. Since

   $$f(x) = (x - 4)(x + 1) = 0$$

   when $x = 4$ and $x = -1$, then we have x-intercepts at $(-1, 0)$ and $(4, 0)$.

   (b) Find any vertical or horizontal asymptotes of f.

   f is a polynomial, so it has no horizontal or vertical asymptotes.

(c) Find the critical points of f, list the intervals where f is increasing and decreasing, and identify any local maximums or minimums.

$$f'(x) = 2x - 3 = 0$$

when $x = 3/2$, so this is the critical values. You can make a table to check where f is increasing and decreasing:

Interval	$(\infty, 3/2)$	$(3/2, \infty)$
Test Point, t	0	2
$f'(t)$	-	$+$
Inc/Dec	Dec	Inc

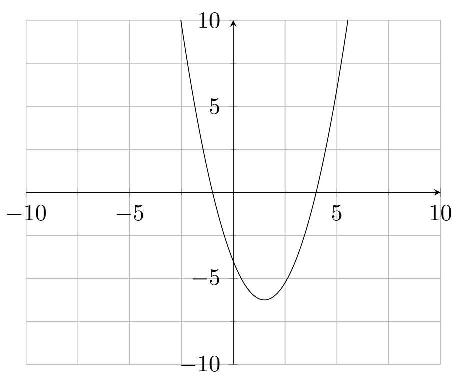
That is, f is decreasing on $(-\infty, 3/2)$ and increasing on $(3/2, \infty)$. $f(3/2) = -25/4$, so we have a critical (stationary) point at $(3/2, -25/4)$, which would be a minimum by the first derivative test.

(d) Find the inflection points of f, list the intervals where f is concave up and concave down.

$$f''(x) = 2$$

which is always positive, so the function is concave up everywhere.

(e) Sketch a graph of f.



For the function $f(x) = \dfrac{2x-6}{4-x}$:

(a) Find any x and y intercepts of f.

Since
$$f(0) = -6/4 = -3/2$$
we have y-intercept $(0, -3/2)$, and since $f(x) = 0$ when $x = 3$, we have an x intercept at $(3, 0)$.

(b) Find any vertical or horizontal asymptotes of f.

The horizontal asymptote would be $y = -2$ (the limit at ∞), and there's a vertical asymptote at $x = 4$ (where the function is undefined).

(c) Find the critical points of f, list the intervals where f is increasing and decreasing, and identify any local maximums or minimums.

$$f'(x) = \frac{2}{(4-x)^2}$$

so $f'(x) \neq 0$ for any x, but $f'(x)$ is undefined at $x = 4$. Use this critical value to set up the intervals to test:

Interval	$(\infty, 4)$	$(4, \infty)$
Test Point, t	0	5
$f'(t)$	$+$	$+$
Inc/Dec	Inc	Inc

That is, f is increasing on $(-\infty, 4)$ and $(4, \infty)$. We have no minimums or maximums.

(d) Find the inflection points of f, list the intervals where f is concave up and concave down.

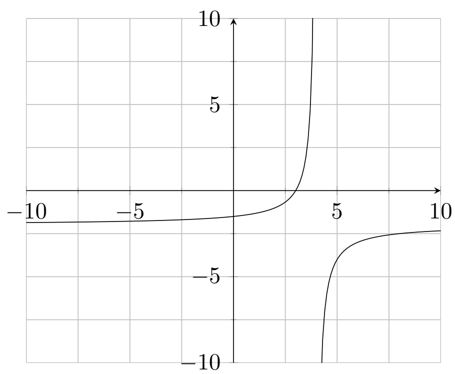
$$f''(x) = \frac{4}{(4-x)^3}$$

The possible inflection points would therefore be at $x = 4$.

Interval	$(-\infty, 4)$	$(4, \infty)$
Test Point, t	0	5
$f''(t)$	$+$	-
Concavity	Up	Down

That is, f is concave up on $(-\infty, 4)$, and concave down on $(4, \infty)$. There is no inflection point, since the function itself is not defined at $x = 4$.

(e) Sketch a graph of f.



4. A cylindrical can is being designed to hold 100 cm^3 of oil. The cost of the can depends only on its surface area. Find the dimensions (height and radius) of the can that will minimize the cost of production. (Hint: You need to find an equation for the surface area of a cylinder. The volume of a cylinder with radius r and height h is$V = \pi r^2 h$; the area of a circle of radius r is πr^2, and the perimeter of circle of radius r is $2\pi r$.)

We need to minimize the surface area of the cylinder with radius r and height h. The top and bottom of the can each have area πr^2. The sides of the can form a rectangle with height h and length $2\pi r$. So the total surface area of the can is given by

$$A = 2\pi r^2 + h(2\pi r)$$

We can't do anything with this expression until we eliminate a variable. The volume of the cylinder must be 100, so

$$\pi r^2 h = 100 \qquad \Rightarrow \qquad h = \frac{100}{\pi r^2}$$

Now

$$A = 2\pi r^2 + \frac{100}{\pi r^2}(2\pi r) = 2\pi r^2 + \frac{200}{r}$$

$$A' = 4\pi r - \frac{200}{r^2} = 0$$

when

$$4\pi r = \frac{200}{r^2} \qquad \Rightarrow \qquad r^3 = 50/\pi$$

so the critical value is $r = \sqrt[3]{50/\pi}$. The height here must be $h = \frac{100}{\pi(\sqrt[3]{50/\pi})^2}$. Note that for this problem, it wouldn't really make sense to "check endpoints". The smallest radius possible is 0, which gives us 0 volume. Similarly, if the radius were as large as possible, then the height would be 0, and again we'd have 0 volume.

5. Let $f(x) = \frac{1}{2}x - \sqrt{x}$. Verify that the hypotheses of Rolle's Theorem are satisfied on the interval $[0, 4]$, and find any values of c predicted by the theorem.

$f(x)$ is a difference of two continuous functions (on this interval, at least) and so it is continuous on $[0, 4]$.
$f'(x) = \frac{1}{2} - \frac{1}{2}x^{-1/2}$, which is also defined for any x on the interval $(0, 4)$, so $f(x)$ is differentiable on that interval.
Finally, note that

$$f(0) = \frac{1}{2}(0) - \sqrt{0} = 0 - 0 = 0$$

and

$$f(4) = \frac{1}{2}(4) - \sqrt{4} = 2 - 2 = 0$$

so $f(0) = f(4)$.
Now all conditions are satisfied, and by Rolle's theorem, there is some c between 0 and 4 so that $f'(c) = 0$. We just need to solve:

$$f'(c) = \frac{1}{2} - \frac{1}{2}c^{-1/2} = 0$$

$$\frac{1}{2} = \frac{1}{2}c^{-1/2}$$

$$2 = 2c^{1/2}$$

and so $c = 1$.

6. Let $f(x) = x^3 + x - 4$. Verify that the hypotheses of the Mean Value Theorem are satisfied on the interval $[-1, 2]$, and find any values of c predicted by the theorem.

$f(x)$ is a polynomial, so it's continuous on $[-1, 2]$ and differentiable on $(-1, 2)$. So the Mean Value Theorem will apply, and it predicts that there is some c between -1 and 2 so that

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{[2^3 + 2 - 4] - [(-1)^3 + (-1) - 4]}{3} = 4$$

Now

$$f'(c) = 3c^2 + 1 = 4$$

when

$$3c^2 = 3 \qquad \Rightarrow \qquad c^2 = 1$$

$$c = \pm 1$$

Only one of these values is *between* -1 and 2, which would be $c = 1$. However, the statement does hold for $c = -1$ in this case. (We don't include the endpoints of the interval when looking for values of c, since we require $f'(c)$ to equal some value, and by hypothesis, $f'(x)$ doesn't even need to be defined at the endpoints of the interval.)

7. Find the integrals

(a) $\int 4x^2 - 5x^3 + 1 \; dx$

$$\int 4x^2 - 5x^3 + 1 \; dx = \frac{4}{3}x^3 - \frac{5}{4}x^4 + x + C$$

Check by taking the derivative:

$$\frac{d}{dx}\left[\frac{4}{3}x^3 - \frac{5}{4}x^4 + x + C\right] = 4x^2 - 5x^3 + 1$$

(b) $\int \sin(x) - \cos(x) \; dx$

$$\int \sin(x) - \cos(x) \; dx = -\cos(x) - \sin(x) + C$$

Check by taking the derivative:

$$\frac{d}{dx}[-\cos(x) - \sin(x) + C] = -(-\sin(x)) - \cos(x) = \sin(x) - \cos(x)$$

(c) $\int x^4 + x^{-4} \; dx$

$$\int x^4 + x^{-4} \; dx = \frac{1}{5}x^5 - \frac{1}{3}x^{-3} + C$$

(d) $\int \frac{4x^8 - 2x^4 + 13x^2}{x^3} \; dx$

Rewrite as a sum and different of fractions, then write using exponents:

$$\int \frac{4x^8 - 2x^4 + 13x^2}{x^3} \; dx = \int \frac{4x^8}{x^3} - \frac{2x^4}{x^3} + \frac{13x^2}{x^3} \; dx = \int 4x^5 - 2x + 13x^{-1} \; dx$$

Now integrate:

$$\int 4x^5 - 2x + 13x^{-1} \; dx = \frac{4}{6}x^6 - x^2 + 13\ln|x| + C = \frac{2}{3}x^6 - x^2 + 13\ln|x| + C$$

(e) $\int \; dx$

$$\int \; dx = x + C$$

(f) $\int (x + \sqrt[3]{x})(2 - x^2)\ dx$

Distribute (expand) first, using exponent properties:

$$\int (x + \sqrt[3]{x})(2 - x^2)\ dx = \int 2x - x^3 + 2x^{1/3} - x^{7/3}\ dx$$

Then integrate:

$$\int 2x - x^3 + 2x^{1/3} - x^{7/3}\ dx = x^2 - \frac{1}{4}x^4 + \frac{3}{2}x^{4/3} - \frac{3}{10}x^{10/3} + C$$

(g) $4\int \sec^2(x)\ dx$

$$4\int \sec^2(x)\ dx = 4\tan(x) + C$$

(h) $\int \frac{5 - 3\sin^2(x)}{\sin^2(x)}\ dx$

Rewrite the integrand as a sum and difference of fractions:

$$\int \frac{5 - 3\sin^2(x)}{\sin^2(x)}\ dx = \int \frac{5}{\sin^2(x)} - \frac{3\sin^2(x)}{\sin^2(x)}\ dx = \int \frac{5}{\sin^2(x)} - 3\ dx$$

Now apply a trig identity, and find the integral:

$$\int \frac{5}{\sin^2(x)} - 3\ dx = \int 5\csc^2(x) - 3\ dx = -5\cot(x) - 3x + C$$

(i) $\int \frac{4}{x^2}\ dx$

$$\int \frac{4}{x^2}\ dx = 4\int x^{-2}\ dx = 4((-1)x^{-1}) + C = -4x^{-1} + C$$

(j) $\int \frac{dx}{1 + x^2}$

$$\int \frac{dx}{1 + x^2} = \tan^{-1}(x) + C$$

(k) $\int \tan(2\theta)\ d\theta$

First, recall that

$$\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)}$$

Now let $u = \cos(2\theta)$, so $du = -2\sin(2\theta)d\theta$ and $-\frac{1}{2}du = \sin(2\theta)d\theta$. Then

$$\int \tan(2\theta)\ d\theta = \int \frac{\sin(2\theta)}{\cos(2\theta)}d\theta = -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln(u) + C = -\frac{1}{2}\ln(\cos(2\theta)) + C$$

(l) $\int x(1+x^3)\ dx$

$$\int x(1+x^3)\ dx = \int x + x^4\ dx = \frac{1}{2}x^2 + \frac{1}{5}x^5 + C$$

8. Find the solution $y(x)$ of the initial value problems:

(a) $y' = 3x^2 - 4, \quad y(0) = 2$

Any function y that satisfies $y' = 3x^2 - 4$ will be of the form

$$y = \int 3x^2 - 4\ dx = x^3 - 4x + C$$

The particular function we need also satisfies $y(0) = 2$, so to find C:

$$2 = y(0) = (0)^3 - 4(0) + C = 0 + C = C$$

which means $C = 2$. So,

$$y = x^3 - 4x + 2$$

(b) $y' = 4x^3 - 9 + 2\sin(x) + 7e^x, \quad y(0) = 15$

Any function y that satisfies $y' = 4x^3 - 9 + 2\sin(x) + 7e^x$ will be of the form

$$y = \int 4x^3 - 9 + 2\sin(x) + 7e^x\ dx = x^4 - 9x - 2\cos(x) + 7e^x + C$$

The particular function we need also satisfies $y(0) = 15$, so to find C:

$$15 = y(0) = (0)^4 - 9(0) - 2\cos(0) + 7e^0 + C = -2 + 7 + C = 5 + C$$

which means $C = 10$. So,

$$y = x^4 - 9x - 2\cos(x) + 7e^x + 10$$