Exam II Review Solutions
Math 170 Calculus I

1. Find a point on the graph of $y = x^2$ where the tangent line to y is parallel to the line $y = 6x - 4$.

   Parallel lines have the same slope, so the tangent line to the curve $y = x^2$ at the x we're looking for has slope 6. This means the derivative of the curve at this x is 6, and since the derivative is $2x$, then $x = 3$.

2. Find $f'(x)$ using the limit definition of the derivative when $f(x) = 3x^2 - 1$. Check your answer using the power rule.

$$\lim_{h\to 0} \frac{3(x+h)^2 - 1 - (3x^2-1)}{h} = \lim_{h\to 0} \frac{3(x^2+2xh+h^2) - 1 - (3x^2-1)}{h}$$
$$= \lim_{h\to 0} \frac{6xh + 3h^2}{h} = \lim_{h\to 0}(6x+3h) = 6x$$

3. Let $f(x) = 4x^2 + 2x - 3$. Find the intervals, if any, where $f(x)$ is increasing. Find the intervals, if any, where $f(x)$ is decreasing.

   We need to find where the derivative is positive, and where it's negative. Since $f'(x) = 8x+2$, then $f'(x)$ is positive whenever $x > -1/4$ and negative elsewhere (I found this by setting $8x+2 = 0$ and solving for x). So, the curve $f(x)$ is increasing when $x > -1/4$ and decreasing when $x < -1/4$. In interval notation, we could say that $f(x)$ is increasing on $(-1/4, \infty)$ and decreasing on $(-\infty, -1/4)$. Why didn't we include $-1/4$ itself?

4. Find the $\frac{dy}{dx}$ (aka y'), using any method you prefer.

   (a) $y = \left(\frac{x^2-2}{2x^2+1}\right)^3$

   Use the chain rule, and the quotient rule. No need to simplify:

$$y' = \frac{(2x^2+1)(2x) - (x^2-2)(4x)}{(2x^2+1)^2} \cdot 3\left(\frac{x^2-2}{2x^2+1}\right)^2$$

   (b) $y = \sin(x^3+2)$

   Use the chain rule:

$$y' = 3x^2\cos(x^3+2)$$

(c) $y = \dfrac{e^x}{\ln(x)}$

Using the quotient rule:

$$y' = \frac{\ln(x)e^x - \frac{e^x}{x}}{\ln(x)^2}$$

(d) $3xy^2 + 6y - 4x^3 = 10$

We really need implicit differentiation - solving for y would be tough. Don't forget to use the product rule when there's an x and y in the same term:

$$3y^2 + 6xyy' + 6y' - 12x^2 = 0$$

$$y'(6xy + 6) = 12x^2 - 3y^2$$

$$y' = \frac{12x^2 - 3y^2}{6xy + 6}$$

(e) $y = e^x \sin^{-1}(x)$

Use the product rule, along with the formula for the derivative of $\sin^{-1}(x)$:

$$y' = e^x \sin^{-1}(x) + e^x \frac{1}{\sqrt{1 - x^2}}$$

(f) $y = \left(\dfrac{\sin(x)}{\cos(x)}\right)^4$

We could use the chain rule, along with the quotient rule. But it's easier if we recognize that $\sin(x)/\cos(x) = \tan(x)$, so $y = \tan^4(x)$. Now we just need the chain rule, and the derivative of $\tan(x)$:

$$y' = 4\tan^3(x)\sec^2(x)$$

(g) $y = \ln(\tan(x))$

Remember that if $y = \ln(f(x))$, then $y' = f'(x)/f(x)$:

$$y' = \frac{\sec^2(x)}{\tan(x)}$$

(h) $y = \dfrac{-2x^3 - 5x}{8x^2 - 10x}$

Simplify first:

$$y = \frac{-2x^2 - 5}{8x - 10}$$

Now use the quotient rule, and don't worry about simplifying:

$$y' = \frac{(8x - 10)(-4x) - (-2x^2 - 5)(8)}{(8x - 10)^2}$$

(i) $y = e^{\sqrt{1-3x^2}}$

$$y' = \frac{d}{dx}(\sqrt{1 - 3x^2})e^{\sqrt{1-3x^2}} = \frac{1}{2}(1 - 3x^2)^{-1/2}(-6x)e^{\sqrt{1-3x^2}}$$

(j) $y = \sin^{-1}(x) + \cos^{-1}(x)$

$$y' = \frac{1}{\sqrt{1 - x^2}} + \frac{-1}{1 - x^2}$$

5. Find y'' if $x^2 + y^2 = 50$. Simplify your answer as much as possible.

First take the derivative with respect to x:

$$2x + 2yy' = 0$$

Then solve for y':

$$y' = \frac{-2x}{2y} = -\frac{x}{y}$$

Now, to find y'', take the derivative of y' with respect to x:

$$y'' = -\frac{y(1) - xy'}{y^2} = \frac{xy' - y}{y^2}$$

We already know what y' is in terms of x and y, so we'll replace it in our formula for y'':

$$y'' = \frac{x(-x/y) - y}{y^2} = \frac{-x^2 - y^2}{y^3}$$

We also know what $x^2 + y^2$ is from the original equation, so

$$y'' = \frac{-(x^2 + y^2)}{y^3} = -\frac{50}{y^3}$$

6. Find y' using logarithmic differentiation if $y = (x^2-5)(3x^4+6)(x^3+1)(3x^4-5x^2+2x)$.

$$\begin{aligned}\ln(y) &= \ln[(x^2-5)(3x^4+6)(x^3+1)(3x^4-5x^2+2x)] \\ &= \ln(x^2-5) + \ln(3x^4+6) + \ln(x^3+1) + \ln(3x^4-5x^2+2x)\end{aligned}$$

Now take the derivative with respect to x (pay attention to the left hand side here!)

$$\frac{y'}{y} = \frac{2x}{x^2-5} + \frac{12x^3}{3x^4+6} + \frac{3x^2}{x^3+1} + \frac{12x^3-10x+2}{3x^4-5x^2+2x}$$

Finally, multiply by y on both sides to find y', keeping in mind that we actually know what y is in this case:

$$\begin{aligned}y' &= y\left(\frac{2x}{x^2-5} + \frac{12x^3}{3x^4+6} + \frac{3x^2}{x^3+1} + \frac{12x^3-10x+2}{3x^4-5x^2+2x}\right) \\ &= (x^2-5)(3x^4+6)(x^3+1)(3x^4-5x^2+2x)\left(\frac{2x}{x^2-5} + \frac{12x^3}{3x^4+6} + \frac{3x^2}{x^3+1} + \frac{12x^3-10x+2}{3x^4-5x^2+2x}\right)\end{aligned}$$

7. Two parallel sides of a rectangle are being lengthened at the rate of 2 in/sec, while the other two sides are shortened in such a way that the figure remains a rectangle with constant area 50 in^2. What is the rate of change of the perimeter of the rectangle when the length of an increasing side is 5 in? Is the perimeter increasing or decreasing?

Let x be the length of a side that's getting longer, and y be the length of a side that's getting shorter. Note that we're given $dx/dt = 2$. The area and perimeter of the rectangle would be

$$50 = xy, \qquad P = 2x + 2y$$

The rate of change of the perimeter would be (taking derivatives with respect to x):

$$\frac{dP}{dt} = 2\frac{dx}{dt} + 2\frac{dy}{dt} = 2(2) + 2\frac{dy}{dt}$$

We need to go back and find dy/dt, using the area formula's derivative:

$$\frac{d}{dt}[50] = x\frac{dy}{dt} + y\frac{dx}{dt}$$

$$0 = x\frac{dy}{dt} + 2y$$

When $x = 5$, we must have $y = 10$, since the area is fixed at 50. Now

$$0 = 5\frac{dy}{dt} + 2(10)$$

so $dy/dt = -4$. Then

$$\frac{dP}{dt} = 4 + 2\frac{dy}{dt} = 4 + 2(-4) = -4$$

So the perimeter is *decreasing* at a rate of 4 in/sec.

8. Find the limits:

(a) $\lim_{x \to +\infty} \frac{x^{60}}{e^x}$

This is type ∞/∞, so we can use L'Hopital's rule:

$$\lim_{x \to +\infty} \frac{x^{60}}{e^x} = \lim_{x \to +\infty} \frac{60x^{59}}{e^x}$$

That didn't help! But we can keep doing it. Eventually, after taking enough derivatives, the numerator will be 0. The denominator never changes, since it's derivative is always just e^x. So after taking 61 derivatives, we get

$$\lim_{x \to +\infty} \frac{x^{60}}{e^x} = \lim_{x \to +\infty} \frac{0}{e^x} = 0$$

(b) $\lim_{x \to +\infty} \frac{5x^3 - 4x + 3}{2x^2 - 1}$

This is also type ∞/∞, so we can use L'Hopital's rule (possibly more than once!):

$$\lim_{x \to +\infty} \frac{5x^3 - 4x + 3}{2x^2 - 1} = \lim_{x \to +\infty} \frac{15x^2 - 4}{4x} = \lim_{x \to +\infty} \frac{30x}{4} = \infty$$

(c) $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin(x)} \right)$

This one is type $\infty - \infty$. We can't use L'Hopital's rule quite yet. We first need to rewrite the expression as something of type 0/0 or ∞/∞. The easiest way (in this case) is to find a common denominator:

$$\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin(x)} \right) = \lim_{x \to 0} \left(\frac{\sin(x) - x}{x \sin(x)} \right)$$

Now we can repeatedly use L'Hopital's rule, stopping whenever we get a limit we can evaluate:

$$\lim_{x \to 0} \left(\frac{\sin(x) - x}{x \sin(x)} \right) = \lim_{x \to 0} \left(\frac{\cos(x) - 1}{x \cos(x) + \sin(x)} \right)$$
$$= \lim_{x \to 0} \left(\frac{-\sin(x)}{x \sin(x) + \cos(x) + \cos(x)} \right) = 0$$